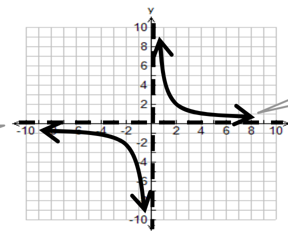
2.10 – Limits

Recall that ***asymptotes*** are imaginary lines (shown as dashed lines) that a function might approach but never actually reaches.

Ex. a) Vertical asymptotes occur because of denominator restrictions
b) Horizontal asymptotes occur when a function seems to tend to a certain value at very large positive or negative values. That is as the domain (x-value) approaches infinity what is the function (y-value) equaling.

We have already looked at vertical asymptotes under the concept of discontinuity, so let us explore the horizontal asymptotes in the following example.

Ex. Given: $f(x)=\frac{1}{x}$



$\frac{1}{-1000}\approx 0$ and $\frac{1}{+1000}\approx 0$

Horizontal asymptotes explore the extreme behaviour of functions. Test large values (i.e. -1000 or +1000) to see if the function is ***limited*** by such an asymptote

Limit is the term used to describe the idea that the function tends to (or is limited to) this certain output (y) value at the considered input (x) value. We use some specific mathematical syntax to deal with this concept.

Ex. Given $f(x)=\frac{1}{x}$ explore horizontal asymptotes using form $\lim_{x\to\infty} f(x)$

Infinity symbol is used for large value(s). As a concept it means both positive and negative. This statement says; what is the limit of f(x) as x approaches infinity

Very large negative numbers

$$\lim_{x\to-\infty}\frac{1}{x}$$
$$=\frac{1}{-\infty}$$
$$=0$$

Normally one can consider both positive and negative behaviour together. I have broken it up here, just to show you how you might think about this.

Recall
?# ÷ small # = big #
?# ÷ big # = small #

So think as follows
?# ÷ small # = ***no asymptote***
?# ÷ big # = ***0***

Very large positive numbers

$$\lim_{x\to+\infty}\frac{1}{x}$$
$$=\frac{1}{+\infty}$$
$$=0$$

Example 1: Determine any horizontal asymptotes using limits form.

a) $f(x)=\frac{8x-3}{x}$

$$=\lim_{x\to\infty}\frac{8x-3}{x}$$
$$=\frac{8(\infty)}{(\infty)}$$
$$=8$$

$\therefore$ asymptote at y = 8

Infinities cancel out. If it helps, just use a large number like 1000.

Taking 3 away from ∞ is insignificant

b) $m(x)=\frac{2x-3}{x^2-9}$

$$=\lim_{x\to\infty}\frac{2x-3}{x^2-9}$$
$$=\frac{2(\infty)}{(\infty)(\infty)}$$
$$=\frac{2}{\infty}$$
$$=0$$

$\therefore$ asymptote at y = 0

c) $g(x)=\frac{2x^3-3x}{x^2-9x}$

$$=\lim_{x\to\infty}\frac{2x^3-3x}{x^2-9x}$$
$$=\frac{2(\infty)^3}{(\infty)^2}$$
$$=2(\infty)$$
$$=\infty$$

$\therefore$ no asymptote

Focus on highest degree terms only as at ∞ what does it matter to subtract ∞^2

d) $h(x)=\sqrt{x-5}$

$$=\lim_{x\to+\infty}\sqrt{x-5}$$
$$=\sqrt{(+\infty)-5}$$
$$=+\infty$$

$\therefore$ no asymptote

Here we should consider one sided (i.e.+∞) limit as domain is restricted to positive values

2.10 – Limits Practice Questions

1. Determine any horizontal asymptotes.

> Questions d,e,f are not rational functions but just considered here to explore limits.

a) $g(x)=\dfrac{2x}{x^2-4x-5}$ b) $g(x)=\dfrac{2x^2}{x^2-4x-5}$ c) $g(x)=\dfrac{2x^3}{x^2-4x-5}$

d) $h(x)=\sqrt{x-3}$ e) $h(x)=\sqrt{5-x}$ f) $h(x)=\sqrt{x^2-5}$

g) $g(x)=\dfrac{2x^2-3}{3x^2-4x-5}$ h) $f(x)=\dfrac{x-2}{x^2-4}$ i) $m(x)=\dfrac{5x+5}{3x+3}$

j) $m(x)=\dfrac{x^2-3}{x^3}$ k) $y=\dfrac{3x}{x^2-7x+12}$ l) $y=\dfrac{(x-3)^2}{x-3}$

2. Evaluate the following limits

a) $\lim\limits_{x\to\infty}\dfrac{9-x^2}{x-3}$ b) $\lim\limits_{x\to\infty}\dfrac{2x-3}{x^2-9}+4$ c) $\lim\limits_{x\to\infty}\dfrac{2x-3}{x-9}+4$

d) $\lim\limits_{x\to\infty}\dfrac{4x-3x^2}{x^2+2}$ e) $\lim\limits_{x\to+\infty}\dfrac{9+5x-2x^2}{4x+3x^2}$ f) $\lim\limits_{x\to+\infty}\dfrac{4-x}{1+\frac{x}{3}}-2$

g) $\lim\limits_{x\to-\infty}\dfrac{-x^2}{x^2}$ h) $\lim\limits_{x\to-\infty}\dfrac{-x}{x}$ i) $\lim\limits_{x\to-\infty}\dfrac{(-x)^2}{x^2}$

3. Draw a possible graph for a function that is discontinuous at x = 2 and has a horizontal asymptote at y = -1.

4. A large tank contains 1000L of pure water. Salt water containing 20g of salt per liter is pumped into the tank at a rate of 10L/min. The concentration of salt, *C(t)* grams per litre, after *t* minutes, is given by the equation $C(t)=\dfrac{20t}{100+t}$

   a) Determine the concentration of salt in the tank after; i) 0h ii) 60min iii) 5h iv) 15h
   b) Using your data from part a, sketch a graph of salt concentration versus time for the first 20 hours.
   c) Determine; $\lim\limits_{t\to\infty}\dfrac{20t}{100+t}$
   d) Interpret the limit in part c in terms of salt in the tank.

Answers 1. a) 0 **b)** 2 **c)** none **d)** none **e)** none **f)** none **g)** 2/3 **h)** 0 **i)** 5/3 **j)** 0 **k)** 0 **l)** none **2. a)** no limit **b)** 4 **c)** 6 **d)** -3 **e)** -2/3 **f)** -5 **g)** -1 **h)** -1 **i)** 1 **4. a) i)** 0g/l **ii)** 7.5g/l **iii)** 15g/l **iv)** 18g/l **c)** 20g/l **d)** the maximum concentration you will ever reach will effectively be 20 grams per litre.