3.7 – Factor Theorem

A division results in a remainder of zero only when the divisor is a factor of the dividend.

Ex. $57 \div 8 = (8)(7) + 1$ but $56 \div 8 = (8)(7) + 0$

> No remainder because 8 is a factor of 56

Using this fact combined with the remainder theorem, one can determine if a given divisor d(x) will be a factor for the given function f(x). ***The factor theorem*** highlights this observation by stating;

(x - p) is a factor of f(x) if and only if $f(p) = 0$

> Note sign change

Example 1: Is $(x – 2)$ a factor of $(x^3 - 3x^2 + 6x – 5)$?

Let $f(x) = x^3 - 3x^2 + 6x – 5$ then test

$$f(2) = (2)^3 – 3(2)^2 + 6(2) – 5 = 8 – 12 + 12 - 5 = 3$$

Since $f(2) \neq 0$ therefore $x – 2$ is **<u>not</u>** a factor

Example 2: Determine any factor(s) of $x^3 + 3x^2 - 16x + 12$

> Basically a trial and error process using different value(s) till you find one that gives a remainder of zero

Test $f(1) = (1)^3 + 3(1)^2 – 16(1) + 12 = 0$ $\therefore x – 1$ is a factor

> Remember to change the sign

Test $f(-1) = (-1)^3 + 3(-1)^2 – 16(-1) + 12 = 26$ $\therefore x + 1$ is **<u>not</u>** a factor

> Although it is trial and error, use some judgement in the value(s) picked based on the coefficients and their signs.

Test $f(2) = (2)^3 + 3(2)^2 – 16(2) + 12 = 0$ $\therefore x - 2$ is a factor

Caring on in this way is impractical. It is enough to find one factor this way. The other factors can be found using synthetic division.

Example 3: Factor $x^3 + 2x^2 - 5x - 6$

> Need to find one factor to start question.

Test $f(-1) = (-1)^3 + 2(-1)^2 – 5(-1) - 6 = 0$ $\therefore x + 1$ is a factor

Now use synthetic division:

-1	1	2	-5	-6
		-1	-1	+6
	1	1	-6	0

> Watch signs. Use same sign as test value here.

> Zero remainder confirms that x + 1 is indeed a factor (double checks your arithmetic)

So we know: $(x^3 + 2x^2 - 5x – 6) = (x + 1)(x^2 + x – 6) = (x + 1)(x + 3)(x – 2)$

> See if you can continue to factor.

> One could always check answers by expanding brackets to get back to original polynomial

3.7 – Factor Theorem Practice Questions

1. Which of the following polynomials have x – 2 as a factor?

a) $x^2 - 3x + 5$ b) $x^3 - 3x^2 - 4x + 12$ c) $x^3 + x^2 - 16x + 20$
d) $-x^3 + 3x - 2$ e) $x^4 - 8x^3 + 24x^2 - 32x + 16$ f) $4x^3 + 32$

2. Determine which binomials listed below are factors for $x^3 - 4x^2 + x + 6$

a) $x - 2$ b) $x + 2$ c) $x - 3$

3. Factor fully

a) $x^3 - 7x - 6$ b) $x^3 + 5x^2 + 2x - 8$ c) $x^3 - 9x^2 + 17x - 6$
d) $2x^3 - x^2 - 13x - 6$ e) $x^3 - 4x + 3$ f) $x^3 + 2x^2 - x - 2$
g) $x^4 - 8x^3 + 3x^2 + 40x - 12$ h) $x^4 - 6x^3 - 15x^2 - 6x - 16$

You will have to use the factor theorem twice - once for a quadric root and once for a cubic root. When one gets to 2nd degree, the quadratic formula can be used to determine factors.

4. Find the value of ***k*** if x + 4 is a factor of $3x^3 + 11x^2 - 6x + k$

5. If (x – 1) is a factor of $x^3 - 2kx^2 + 3x + 1$, what is the value of ***k***?

6. We know that (x – y) is a factor of $x^3 - y^3$ because $f(y) = (y)^3 - y^3 = 0$.

The other factor, $(x^2 + xy + y^2)$, can be determined through long division.

Perform the same divisions on the following to come up with a pattern to help you factor the questions that follow the chart.

Expression	1st factor	2nd factor
$x^2 - y^2$	$x - y$	$x + y$
$x^3 - y^3$	$x - y$	$x^2 + xy + y^2$
$x^4 - y^4$	$x - y$	
$x^5 - y^5$	$x - y$	
$x^n - y^n$	$x - y$	

The general case to factor any binomial differences

a) $x^4 - 81$ b) $x^5 - 32$ c) $8x^3 - 125$ d) $64x^6 - 1$

Answers 1. b, c, e **2.** a, c **3. a)** (x+1)(x+2)(x-3) **b)** (x-1)(x+4)(x+2) **c)** $(x-2)(x^2-7x+3)$ **d)** (2x+1)(x+2)(x-3)
e) $(x-1)(x^2+1x-3)$ **f)** (x+1)(x+2)(x-1) **g)** $(x-3)(x+2)(x^2-7x+2)$ **h)** $(x+2)(x-8)(x^2+1)$ **4.** k = -8 **5.** k=2.5
6. pattern is $x^n-y^n=(x-y)(x^{n-1}+x^{n-2}y+\ldots+xy^{n-2}+y^{n-1})$ **a)** $(x-3)(x^3+3x^2+9x+27)$ **b)** $(x-2)(x^4+2x^3+4x^2+8x+16)$
c) $(2x-5)(4x^2-10x+25)$ **d)** $(2x-1)(32x^5+16x^4+8x^3+4x^2+2x+1)$