4.4 – Applications of the Exponential Function

Many natural processes that involve growth or decomposition (decay) can be modelled using the exponential functions. Below are some examples.

**Example 1:** A population of bacteria doubles every 4 hours. Determine an equation to model the bacteria growth as a function of time.

<table>
<thead>
<tr>
<th>Generation</th>
<th>Time (h)</th>
<th># of bacteria (using actual numbers)</th>
<th># of bacteria (in exponential form)</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>x</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>2x</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>4x</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>12</td>
<td>8x</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>16</td>
<td>16x</td>
<td>16</td>
<td></td>
</tr>
</tbody>
</table>

Therefore exponential as a function of time would be: \( P(t) = x2^{\frac{t}{4}} \) with \( t \) in hours

**Example 2:** If human population doubles every 35 years, estimate the World population in 2050 given population in 1999 was about 6B.

- **Know the following:** base = 2 (doubling)
- **Know the following:** time interval = 35
- **Know the following:** year adjustment = \( y - 1999 \)

Therefore exponential model:

\[
P(y) = 6 \cdot 2^{\frac{y - 1999}{35}}
\]

So in 2050:

\[
P(2050) = 6 \cdot 2^{\frac{2050 - 1999}{35}} = 16.5B
\]

**Example 3:** The half life of radium is 1620 years. If have 10mg of radium today, how much will one have in 50 years?

- **Know the following:** base = \( \frac{1}{2} \) (halving each time)
- **Know the following:** original amount = 10
- **Know the following:** time interval = 1620

Therefore exponential model:

\[
A(t) = 10 \cdot \left( \frac{1}{2} \right)^{\frac{t}{1620}}
\]

So in 50 years:

\[
A(50) = 10 \cdot \left( \frac{1}{2} \right)^{\frac{50}{1620}} = 9.79mg
\]

In general:

\[
A_f = A_o (base)^{\frac{t}{t_{half}}}
\]

where

- \( A_f \) = final amount at end
- \( A_o \) = original amount
- base = \( \frac{1}{2} \) if decay, 2 if doubling, or ? to suit situation
- \( t \) = elapsed time
\[ t_i = \text{time interval it takes to double, half-life or ?} \]
4.4 – Applications of the Exponential Function Practice Questions

1. In 1930 a town’s population was 125000. In 1992 it was 500000. Estimate population in 2020.

Know:  \[ 500000 = 125000 \cdot B^{(1992 - 1930)} \]
\[ 4 = B^{62} \]
\[ B = 1.0226 \]

So:  \[ P(y) = 125000 \cdot (1.0226)^{(y - 1930)} \]

Then:  \[ P(2020) = 125000 \cdot (1.0226)^{(2020 - 1930)} \]
\[ = 935122 \]

Therefore town population in 2020 should be around 935000

2. Light intensity is reduced by 2.5% for every meter of water it penetrates into. If we start with 100% light at the surface then  \[ I(d) = 100 \cdot (0.975)^d \], where \( d \) is depth of water in meters.

a) Why is the base 0.975 and not 1.025?
b) What is light intensity at 5m, 10m, 15m, 20m, 25m?
c) When will light intensity be less than 1%?

3. A bamboo plant increases its height by 12% every day. How tall will the plant be after 4 weeks given it starts at a height of 40 cm?

4. Given that a ball bounces to 70% of its previous height on each successive bounce, come up with an exponential function to model the ball height as a function of the number of bounces, where \( h_0 \) is initial height dropped from. Use your equations to find the ball’s height after 7 bounces if dropped from 25 meters.

5. If paid an interest rate of 8%/a compounded annually, come up with an equation to model the total amount as a function of \( n \) years. Use an exponential model to determine total amount in your account after 15 years on a $5000 investment.

6. The half life of carbon-14 (C\(_{14}\)) is about 5760. If a bone sample only has 30% C\(_{14}\) remaining, determine the age of the bone.

7. A helium balloon loses 10% of its volume every 6 hours. What percentage of the original volume will be remaining after 27 hours?

8. A original sample of 160 mg of radioactive sodium (Na\(_{24}\)) was reduced to 20 mg after 45h.

a) Determine half-life of Na\(_{24}\)?
b) Model Na\(_{24}\) remaining as a function of time.
c) If you require 100 mg of Na\(_{24}\) 12h from now, how much will you need now?

Answers 2. a) because reducing by 2.5 means 97.5=100-2.5 remains b) 88%, 78%, 68%, 60%, 53% c) around 181m 3. H(t)=40(1.12), 9.55m 4. H(b)=h(0.70)^b, 2.06m 5. A(y)=A(0.8)^y, $15860.85 6. A=100(1/2)^{y/5760}$, $10004a 7. V=100(0.90)^{y/6}$, 62% 8. a) 15h b) A=160(1/2)^y, $74mg

4.4 – applications exponential function