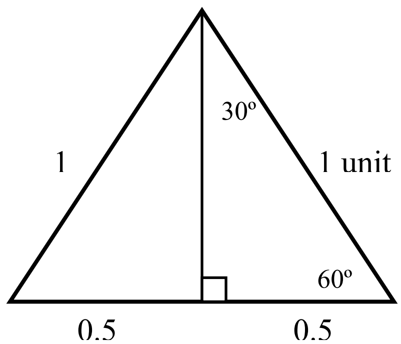
5.3 – Special Triangles and the Unit Circle

Since certain angles (0º, 30º, 45º, 60º, 90º, 180º, …) are very commonly used, it was beneficial to come up with exact trigonometric ratios for these angles. To come up with these exact ratios one can consider an equilateral triangle whose sides are all 1 unit and all angles are 60º.



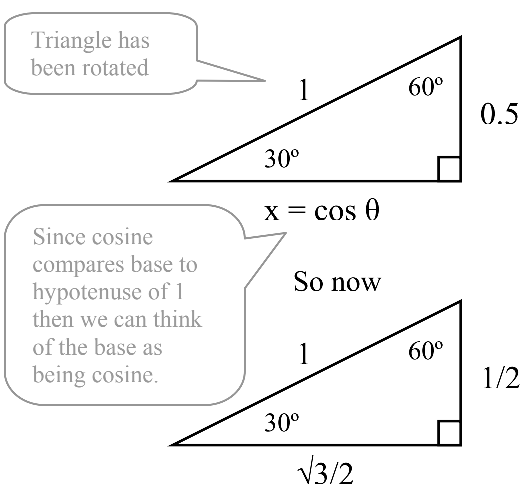
Also recall the basic trigonometric ratios in a right triangle

$\sin\theta = \frac{opp}{hyp}$ compares height of triangle to hypotenuse

$\cos\theta = \frac{adj}{hyp}$ compares base of triangle to hypotenuse

$\tan\theta = \frac{opp}{adj}$ compares height of triangle to its base

Using these principles, along with the Pythagorean Theorem, one can start to determine the exact trig ratios for 30º using half the above triangle.



To start set up Pythagorean relationship to calculate side x

$$x^2 = 1^2 - \left(\frac{1}{2}\right)^2$$

$$x^2 = 1 - \frac{1}{4}$$

$$x^2 = \frac{3}{4}$$

$$x = \frac{\sqrt{3}}{2}$$

Then set up your basic ratios.

$$\sin 30^\circ = \frac{\frac{1}{2}}{1} \quad or \quad \sin 30^\circ = \frac{1}{2}$$

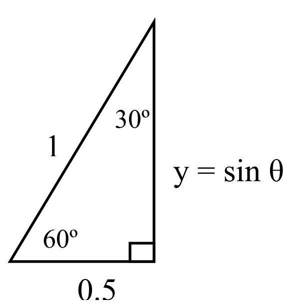
$$\cos 30^\circ = \frac{\frac{\sqrt{3}}{2}}{1} \quad or \quad \cos 30^\circ = \frac{\sqrt{3}}{2}$$

$$\tan 30^\circ = \frac{1}{2} \div \frac{\sqrt{3}}{2}$$

$$\tan 30^\circ = \frac{1}{2} \times \frac{2}{\sqrt{3}}$$

$$\tan 30^\circ = \frac{1}{\sqrt{3}}$$

Moving on to 60º trigonometric ratios one can set up the following;



Using the same logic one can see that y = √3/2

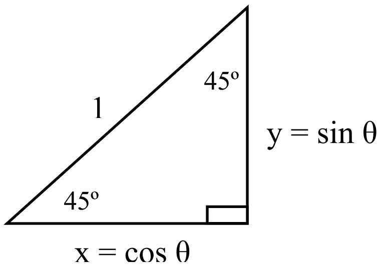
So we get the following ratios; $\sin 60^\circ = \frac{\sqrt{3}}{2}$ $\cos 60^\circ = \frac{1}{2}$ $\tan 60^\circ = \sqrt{3}$

Or in radians; $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ $\cos\frac{\pi}{3} = \frac{1}{2}$ $\tan\frac{\pi}{3} = \sqrt{3}$

One can also notice the complimentary property of sine and cosine ratios. Namely

$$\sin\theta^\circ = \cos(90 - \theta^\circ) \quad or \quad \sin(90 - \theta^\circ) = \cos\theta^\circ$$

Lastly the 45º trigonometric rations can be determined using a 45º triangle.



The following Pythagorean relationship can be set up;

$$1^2 = x^2 + y^2$$
$$1^2 = 2x^2$$
$$x^2 = \frac{1}{2}$$
$$x = \frac{1}{\sqrt{2}}$$

Since y=x in this triangle

Can then set up the following ratios

$\sin 45^\circ = \frac{1}{\sqrt{2}}$ $\quad \cos 45^\circ = \frac{1}{\sqrt{2}}$ $\quad \tan 45^\circ = 1$

$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ $\quad \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ $\quad \tan\frac{\pi}{4} = 1$

The above triangle also lets one set up the Pythagorean identity.

$$y^2 + x^2 = 1 \quad or \quad \sin^2\theta + \cos^2\theta = 1$$

So far we have exact trigonometric ratios for 30º, 45º and 60º. To generate exact trigonometric ratios for 0º, 90º as well as corresponding ratios for 30º, 45º and 60º as one rotates through 360º we can place the hypotenuse of unit length as a radial (i.e. radius) arm that will rotate though the entire circle. Rotating this radial arm generates what is referred to as a "Unit Circle". See attached page for the details on this circle.

Although the "Unit Circle" is probably the easiest way to remember the all the exact ratios, one could also use the special triangles and CAST system to get the corresponding ratios as you rotate around 360º.

In summary we now have the following;

OR in RADIANS

$\sin 0^\circ = 0$	$\cos 0^\circ = 1$	$\tan 0^\circ = 0$	$\sin 2\pi = 0$	$\cos 2\pi = 1$	$\tan 2\pi = 0$
$\sin 30^\circ = \frac{1}{2}$	$\cos 30^\circ = \frac{\sqrt{3}}{2}$	$\tan 30^\circ = \frac{1}{\sqrt{3}}$	$\sin\frac{\pi}{6} = \frac{1}{2}$	$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$	$\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$
$\sin 45^\circ = \frac{1}{\sqrt{2}}$	$\cos 45^\circ = \frac{1}{\sqrt{2}}$	$\tan 45^\circ = 1$	$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$	$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$	$\tan\frac{\pi}{4} = 1$
$\sin 60^\circ = \frac{\sqrt{3}}{2}$	$\cos 60^\circ = \frac{1}{2}$	$\tan 60^\circ = \sqrt{3}$	$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$	$\cos\frac{\pi}{3} = \frac{1}{2}$	$\tan\frac{\pi}{3} = \sqrt{3}$
$\sin 90^\circ = 1$	$\cos 90^\circ = 0$	$\tan 90^\circ = n/a$	$\sin\frac{\pi}{2} = 1$	$\cos\frac{\pi}{2} = 0$	$\tan\frac{\pi}{2} = n/a$

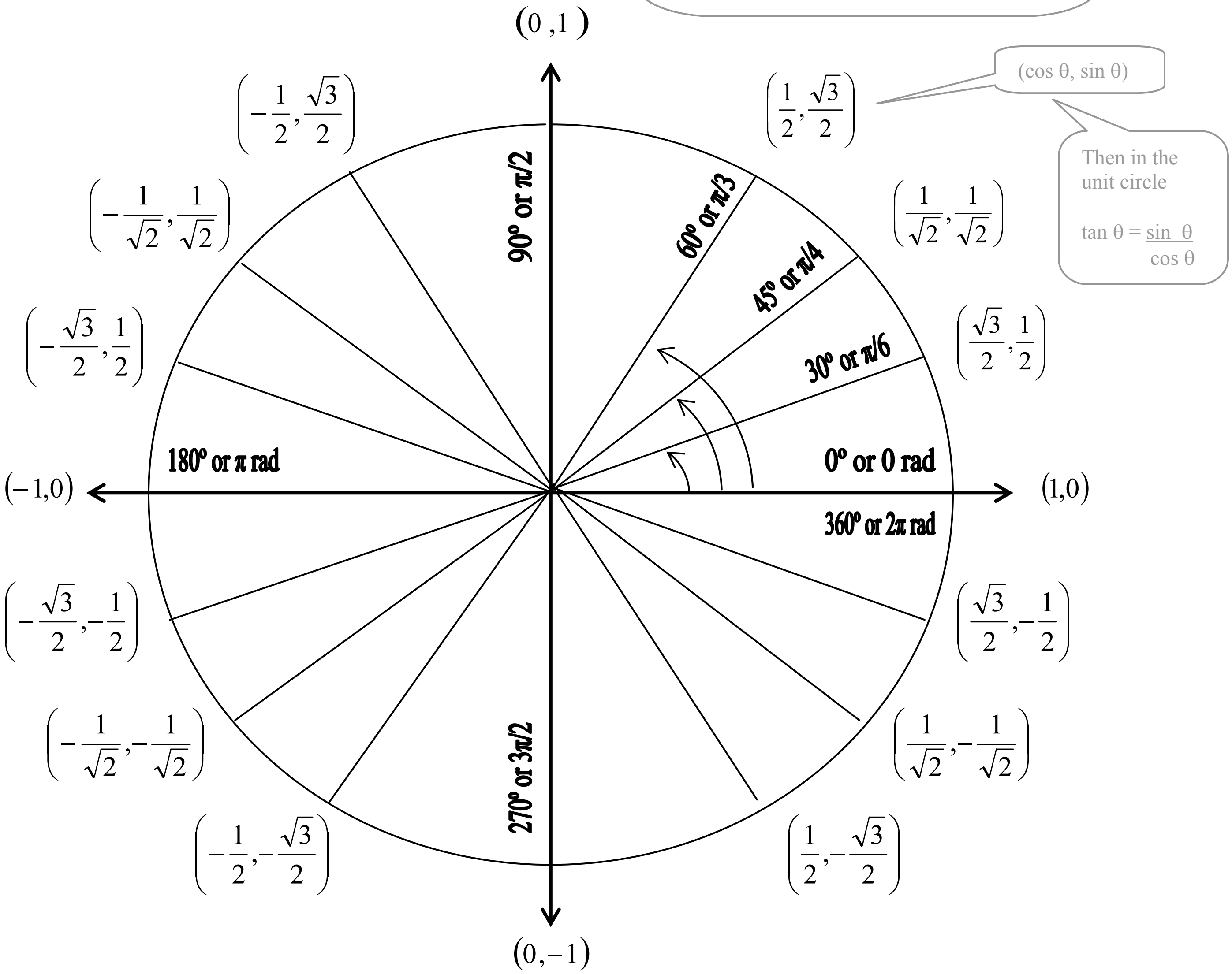
In this modern day of cheap and accessible calculators it is also useful to memorize the decimal equivalent ratios so that calculated answers can be converted to exact ones;

$0 = 0 \qquad 0.5 = \frac{1}{2} \qquad 0.57 = \frac{1}{\sqrt{3}} \qquad 0.70 = \frac{1}{\sqrt{2}} \qquad 0.86 = \frac{\sqrt{3}}{2} \qquad 1=1 \qquad 1.71 = \sqrt{3}$

So when you see this result on your calculator, convert to the corresponding exact ratio

5.3 - Unit Circle Investigation

The benefit of using a radial arm of length 1 unit means that our ratios become actual (x,y) coordinates. One can then memorize these points without the extra complication of using the CAST memory aid. Furthermore the 'Unit Circle" allows us to generate a useful chart to plot the basic trigonometric ratios.



θº	0º	30º	45º	60º	90º	120º	135º	150º	180º	210º	225º	240º	270º	300º	315º	330º	360º
θº (in rads)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$											
Sin θº	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1												
Cos θº	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0												
Tan θº	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	-												
Pt (x,y) (exact)	(1,0)	$\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$	$\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$	$\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$													
Pt (x,y) (decimal)	(1,0)	(0.87,0.5)	(0.7,0.7)	(0.5,0.87)													

Example 1: Find exact trigonometric ratios for the following

a) sin 60°

Calculator = 0.866

So equivalent = √3/2

or

Using unit circle just read the value

$$\sin 60^\circ = \frac{\sqrt{3}}{2}$$

b) $\csc\frac{\pi}{4}$

$$\csc\frac{\pi}{4} = \frac{1}{\sin\frac{\pi}{4}}$$

$$= \frac{1}{\frac{1}{\sqrt{2}}}$$

$$= 1 \div \frac{1}{\sqrt{2}}$$

$$= 1 \times \frac{\sqrt{2}}{1}$$

$$= \sqrt{2}$$

c) tan 120°

Calculator = -1.7

so write = -√3

d) $\cos\frac{11\pi}{6}$

$$\cos\frac{11\pi}{6} = \frac{\sqrt{3}}{2}$$

e) $\cot\frac{\pi}{2}$

$$\cot\frac{\pi}{2} = \frac{1}{\tan\frac{\pi}{2}}$$

$$= 1 \div \tan\frac{\pi}{2}$$

$$= 1 \div \textit{undefined}$$

$$= \textit{undefined}$$

f) $\sec\frac{\pi}{6}$

$$\sec\frac{\pi}{6} = \frac{1}{\cos\frac{\pi}{6}}$$

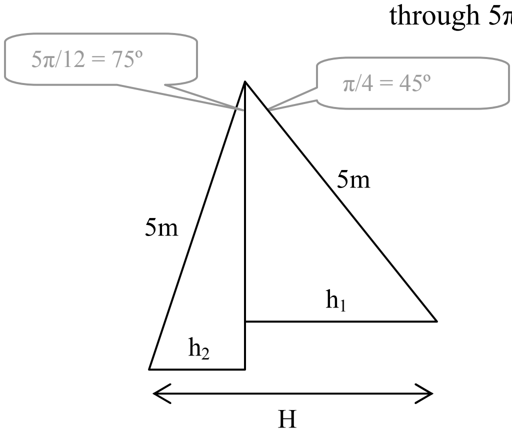
$$= \frac{1}{} \div \cos\frac{\pi}{6}$$

$$= \frac{1}{} \div \frac{\sqrt{3}}{2}$$

$$= \frac{1}{} \times \frac{2}{\sqrt{3}}$$

$$= \frac{2}{\sqrt{3}} \quad or \quad \frac{2\sqrt{3}}{3}$$

Example 2: Find exact total horizontal displacement when a 5 meter long swing rotates through 5π/12 radians, if it starts π/4 left of vertical.



$h_1 = 5\sin 45^\circ$ $\quad$ $h_2 = 5\sin 30^\circ$ (30º=75º-45º)

$H_{total} = 5\sin 45^\circ + 5\sin 30^\circ$ $\quad$ so

$$h_t = 5\left(\frac{1}{\sqrt{2}} + \frac{1}{2}\right)$$

$$h_t = 5\left(\frac{\sqrt{2}}{2} + \frac{1}{2}\right)$$

$$h_t = 2.5(\sqrt{2} + 1)$$

∴ Swing displaces about 12.1 m horizontally.

5.3 – Special Triangles and the Unit Circle Practice Questions

1. Use the accompanying sheet to complete the unit circle from memory.

2. Find exact trigonometric ratio for the following

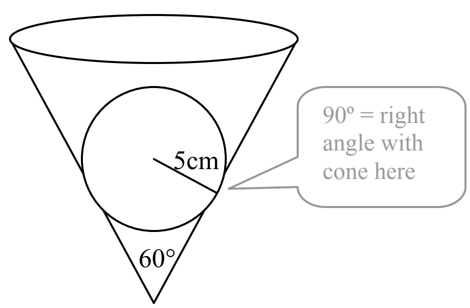
a) sec 30º	b) tan $2\pi/3$	c) sin $\pi/6$	d) sin $\pi/3$
e) csc 45º	f) cos $11\pi/6$	g) csc 2π	h) cot 540º
i) sin $-3\pi/4$	j) cos 300º	k) cot 60º	l) cot $-5\pi/6$
m) cos $7\pi/6$	n) csc $3\pi/2$	o) sec 45º	p) cos $-4\pi/3$
q) sin 1200º	r) tan 5π	s) tan $-7\pi/2$	t) csc 30º
u) csc $\pi/4$	v) cot -30º	w) csc 450º	x) cos 270º

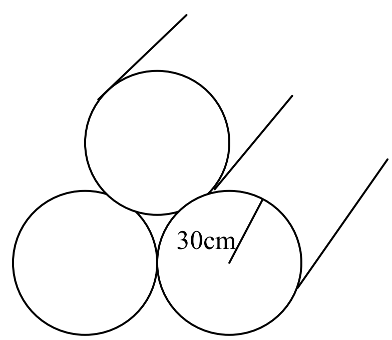
3. A car gate has a 12m long arm. If the gate is lowered from 45° to 30°, calculate the vertical displacement at the end of the arm as an exact value.

4. A cargo ship is tied up at a dock. At low tide a 12m long unloading ramp slopes down from the ship to the dock and makes an angle of 30° to the horizontal. At high tide the ramp makes an angle of 45° to the horizontal.

   a) Determine the change in the horizontal distance from the ship to the dock between low and high tide.
   b) Determine the change in the height of the upper end of the ramp between low and high tide.

5. The vertical angle of a cone is 60° as shown in diagram below. A sphere of radius 5cm is dropped into the cone. How far is the centre of the sphere from the vertex of the cone?

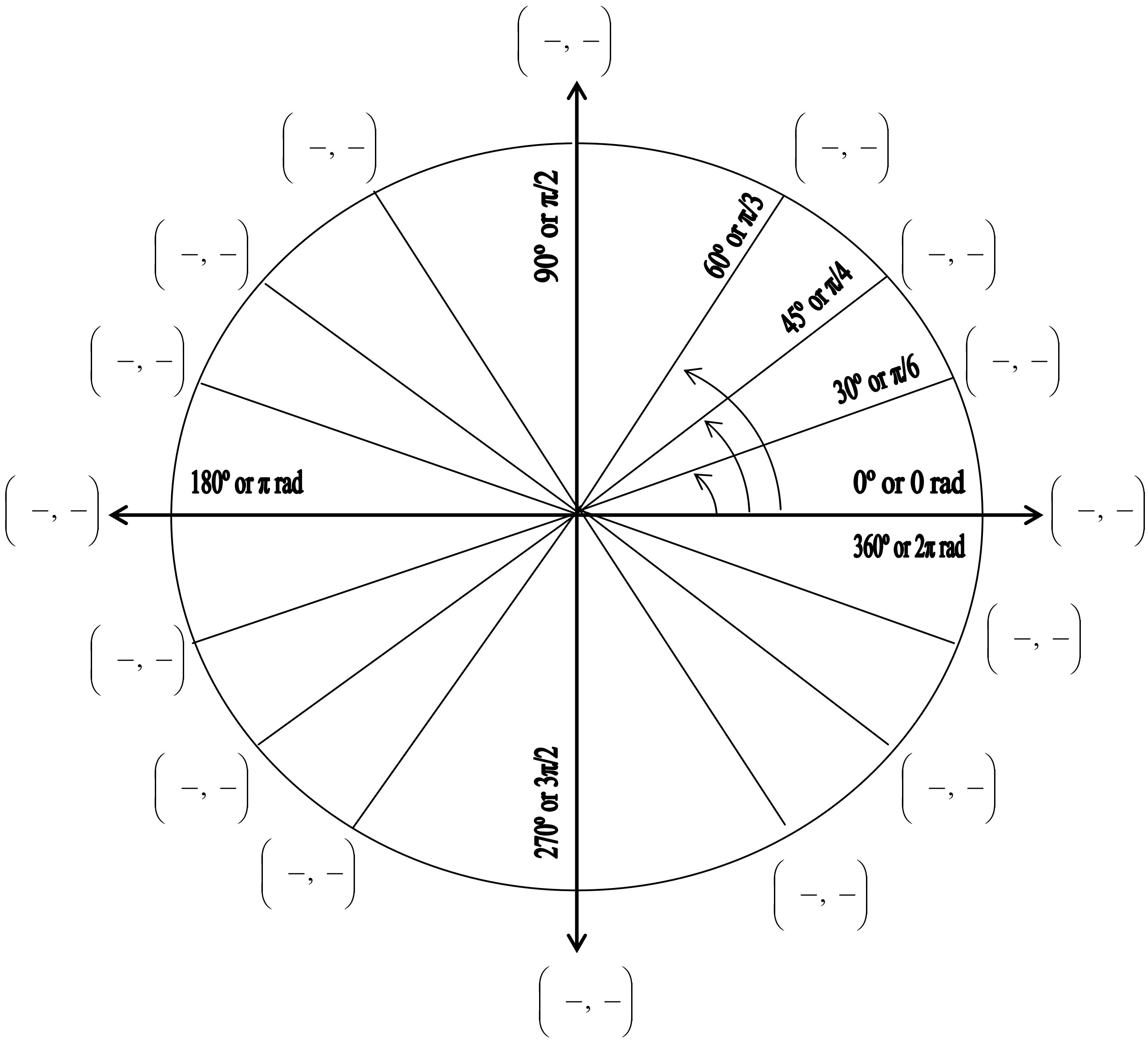




6. Three cylindrical logs of radius 30cm are piled as shown above. How far from the ground is the top of the upper log?

Answers 2. a) $2/\sqrt{3}$ **b)** $-\sqrt{3}$ **c)** 1/2 **d)** $\sqrt{3}/2$ **e)** $\sqrt{2}$ **f)** $\sqrt{3}/2$ **g)** undefined **h)** $\sqrt{3}$ **i)** -1/2 **j)** 1/2 **k)** $1/\sqrt{3}$ **l)** $\sqrt{3}$ **m)** $-\sqrt{3}/2$ **n)** -1 **o)** $\sqrt{2}$ **p)** -1/2 **q)** $\sqrt{3}/2$ **r)** 0 **s)** undefined **t)** 2 **u)** $\sqrt{2}$ **v)** $-\sqrt{3}$ **w)** 1 **x)** 0 **3. a)** $6(\sqrt{2}-1)$m **b)** 2.5m **4. a)** $6(\sqrt{3}-\sqrt{2})$m or 1.9m **b)** $6(\sqrt{2}-1)$m or 2.5m **5.** 10cm **6.** $30(2+\sqrt{3})$cm or 112cm

5.3 - Unit Circle Investigation



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Sin θº	0				1												
Cos θº	1				0												
Tan θº	0				-												
Pt (x,y) (exact)	(1,0)																
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