

5.5A – Compound Angle Formulas (Sum Identities)

Recall the special (common) angles and their exact ratios that can be generated from the special triangles or the Unit circle.

| x° | 0° or 2π | 30° or $\pi/6$ | 45° or $\pi/4$ | 60° or $\pi/3$ | 90° or $\pi/2$ |
|-----------|---------------------|-----------------------|-----------------------|-----------------------|-----------------------|
| x | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ |
| Sin x | 0 | $\frac{1}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{\sqrt{3}}{2}$ | 1 |
| Cos x | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{2}$ | 0 |
| Tan x | 0 | $\frac{1}{\sqrt{3}}$ | 1 | $\sqrt{3}$ | - |

Using the Unit Circle one can recognize some basic relationships or identities.

Ex. Co-function relationship $\sin\left(\frac{\pi}{2} - x\right) = \cos x$ and $\cos\left(\frac{\pi}{2} - x\right) = \sin x$

Complimentary

Supplemental relationship $\sin \theta^\circ = \sin(\pi - \theta^\circ)$ and $\cos \theta^\circ = -\cos(\pi - \theta^\circ)$

Positional relationship $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$

So negative angle will give the same ratio as positive angle. Just need to adjust for sign

Further development using some basic geometry allows one to also come up with the following formulas for adding and subtracting angles within a trigonometric function

Ex. $\cos(x + y) = \cos x \cos y - \sin x \sin y$ and $\sin(x + y) = \sin x \cos y + \cos x \sin y$
 $\cos(x - y) = \cos x \cos y + \sin x \sin y$ and $\sin(x - y) = \sin x \cos y - \cos x \sin y$

These compound (i.e. one angle made up from two other angles) formulas allows one to find exact trigonometric ratios for angles other than the common ones listed in table above.

Ex. a) $15^\circ = 45^\circ - 30^\circ$ so $\cos(15^\circ) = \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$

b) $\frac{2\pi}{3} = \frac{\pi}{2} + \frac{\pi}{6}$ so $\sin\left(\frac{\pi}{2} + \frac{\pi}{6}\right) = \sin \frac{\pi}{2} \cos \frac{\pi}{6} + \cos \frac{\pi}{2} \sin \frac{\pi}{6}$

Verify this formula on your calculator by comparing answers to both sides.

Example 1: Evaluate the following as exact answers

a) $\cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \cos \frac{\pi}{3} \cos \frac{\pi}{4} + \sin \frac{\pi}{3} \sin \frac{\pi}{4}$
 $= \frac{1}{2} \cdot \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}}$
 $= \frac{1 + \sqrt{3}}{2\sqrt{2}}$
 $= \frac{\sqrt{2} + \sqrt{6}}{4}$

Rationalize the denominator for final answer

b) $\sin\left(\frac{5\pi}{12}\right) = \sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right)$
 $= \sin \frac{\pi}{4} \cos \frac{\pi}{6} + \cos \frac{\pi}{4} \sin \frac{\pi}{6}$
 $= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$
 $= \frac{\sqrt{3} + 1}{2\sqrt{2}}$
 $= \frac{\sqrt{6} + \sqrt{2}}{4}$

Re-write compound angle as sum or difference of two known exact ratios

5.5A –Compound Angle Formulas Practice Questions

- Expand the following to express each as values of trigonometric functions of one number.
 - $\cos(x - 3)$
 - $\sin(y - 4)$
 - $\cos(2x + 3y)$
 - $\sin(2x + 3y)$
- Use compound formula to express each of the following as a single trigonometric functions.
 - $\cos 2x \cos x - \sin 2x \sin x$
 - $\sin 5a \cos 2a - \cos 5a \sin 2a$
 - $\sin 3x \cos x + \cos 3x \sin x$
 - $\cos x \cos x - \sin x \sin x$
- Given $\frac{5\pi}{12} = \frac{\pi}{4} + \frac{\pi}{6}$ Find $\cos \frac{5\pi}{12}$ as exact value
- Given $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$ Find $\cos \frac{\pi}{12}$ as exact value
- Given $\frac{2\pi}{3} = \frac{\pi}{2} + \frac{\pi}{6}$ Find $\sin \frac{2\pi}{3}$ as exact value
- Given $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$ Find $\sin \frac{\pi}{12}$ as exact value
- Given $\frac{2\pi}{3} = \frac{\pi}{2} + \frac{\pi}{6}$ Find $\cos \frac{2\pi}{3}$ as exact value
- Evaluate each of the following as an exact answer.
 - $\cos \frac{5\pi}{6}$
 - $\sin \frac{4\pi}{3}$
 - $\cos \frac{3\pi}{4}$
 - $\sin \frac{5\pi}{4}$
 - $\sin 15^\circ$
 - $\cos \frac{5\pi}{4}$
 - $\cos 75^\circ$
 - $\sin \frac{5\pi}{6}$

Answers 1. a) $\cos x \cos 3 + \sin x \sin 3$ **b)** $\sin y \cos 4 - \cos y \sin 4$ **c)** $\cos 2x \cos 3y - \sin 2x \sin 3y$

d) $\sin 2x \cos 3y + \cos 2x \sin 3y$ **2. a)** $\cos 3x$ **b)** $\sin 4x$ **c)** $\sin 3a$ **d)** $\cos 2x$ **3)** $\frac{\sqrt{6} - \sqrt{2}}{4}$ **4)** $\frac{\sqrt{6} + \sqrt{2}}{4}$ **5)** $\frac{\sqrt{3}}{2}$
6) $\frac{\sqrt{3} - 1}{2\sqrt{2}}$ **7)** $-\frac{1}{2}$ **8. a)** $-\frac{\sqrt{3}}{2}$ **b)** $-\frac{\sqrt{3}}{2}$ **c)** $-\frac{1}{\sqrt{2}}$ **d)** $-\frac{1}{\sqrt{2}}$ **e)** $\frac{\sqrt{6} - \sqrt{2}}{4}$ **f)** $-\frac{1}{\sqrt{2}}$ **g)** $\frac{\sqrt{6} - \sqrt{2}}{4}$ **h)** $\frac{1}{2}$